The Decomposition Method for Studying a Higher-Order Nonlinear Schrödinger Equation in Atmospheric Dynamics

Zheng-Yi Ma

Department of Mathematics, Zhejiang Lishui University, Zhejiang Lishui 323000, P. R. China
Shanghai Institute of Applied Mathematics and Mechanics, Shanghai University, Shanghai 200072, P. R. China

Reprint requests to Dr. Z.-Y. M.; E-mail: mazhengyi_77@yahoo.com.cn

Z. Naturforsch. **62a,** 387 – 395 (2007); received January 22, 2007

The Adomian decomposition method is implemented for solving a higher-order nonlinear Schrödinger equation in atmospheric dynamics. By means of Maple, the Adomian polynomials of an obtained series solution have been calculated. The results reported in this paper provide further evidence of the usefulness of Adomian decomposition for obtaining solutions of nonlinear problems.

Key words: Adomian Decomposition Method; Higher-Order Nonlinear Schrödinger Equation; Adomian Polynomials.

1. Introduction

With the help of a multi-scale expansion method, Lou derived a type of the higher-order nonlinear Schrödinger equation from the nondimensional barotropic and quasi-geostrophic potential vorticity equation without forcing and dissipation on a beta-plane channel to describe nonlinear modulated Rossby waves [1]. This higher-order nonlinear Schrödinger equation includes fourth-order dispersion with a parabolic nonlinearity law, which is one of the most important models in the study of atmospheric and ocean dynamical systems. The high dispersive cubic-quintic nonlinear Schrödinger (HDCNS) equation can be written in the form

$$\begin{aligned}&\mathrm{i}\Psi_z+\lambda\Psi_{tt}+\delta|\Psi|^2\Psi\\&-\mathrm{i}\varepsilon\left[R_1\Psi_{ttt}+R_2(|\Psi|^2\Psi)_t+R_3\Psi(|\Psi|^2)_t\right]\\&+\varepsilon^2[R_4\Psi_{tttt}+R_5(|\Psi|^2\Psi)_{tt}+R_6(\Psi(|\Psi|^2)_t)_t\\&\qquad+R_7(\Psi_t|\Psi|^2)_t\big]=0,\end{aligned} \tag{1}$$

with the initial condition

$$\Psi(t,0)=f(t),$$

where $\mathrm{i}=\sqrt{-1}$, $\Psi\equiv\Psi(z,t)$, t represents the time (in the group-velocity frame), z represents the distance along the direction of propagation (the longitudinal coordinate), and the coefficients λ,δ,R_i $(i=1,2,\cdots,7)$ are constants. Equation (1) without the terms $O(\varepsilon^2)$ has been derived by Dysthe from the nonlinear deep water waves [2], and by Kodama and Hasegawa from the optical communications, respectively [3], and its solutions have been discussed.

It is worthwhile to investigate the numerical solutions (in particular soliton solutions) for the HDCNS equation. In this paper, we aim to introduce a reliable algorithm, the Adomian decomposition method (ADM), to approach (1) with initial profile. Until now, several studies have been conducted to implement the ADM to the nonlinear Schrödinger (NLS) equation [4 – 7]. ADM is a numerical technique for solving a wide class of linear or nonlinear, algebraic or ordinary/partial differential equations. The method, which is well addressed in [8, 9], has a useful attraction in that it provides the solution as an infinite series in which each term can be easily determined. The series is quickly convergent towards an accurate solution. It has been proved to be a competitive alternative to the Taylor series method and other series techniques. Several papers deal with the comparison of the ADM to some existing techniques in solving different types of problems. It was found that, unlike other series solution methods, the decomposition method is easy to program in engineering problems, and provides immediate and visible solution terms without linearization and discretization. Advantages of the ADM over the Picard's method have been proved by Rach [10]. He showed that the two methods are not the same and the Picard's method works only if the equation satisfies the Lipschitz condition. Edwards et al. [11] have introduced their comparison of the ADM and Runge-Kutta method for approximate solutions of some predator prey model equations. Wazwaz introduced a comparison between the ADM and Taylor series method [12]; he showed that the ADM minimizes the computational difficulties of the Taylor series in that the components of the solution are determined elegantly by using simple integrals, although the Taylor series method provides the same answer obtained by the ADM. In [13], the ADM and wavelet-Galerkin method is compared. From the computational viewpoint, the comparison shows that the ADM is efficient and easy to use. In [14], a comparison of the numerical results is obtained by using the B-spline finite element method and ADM. From the results, the ADM algorithm provides highly accurate numerical solutions without spatial discretizations for the nonlinear partial differential equation. The illustrations show that the ADM is numerically more accurate than the conventional numerical method of the finite element. Subsequent works in this direction have demonstrated the power of the method for numerical evaluations.

2. The Method of Solution

This section is devoted to review the ADM for solving the HDCNS equation with the initial condition $\Psi(t,0) = f(t)$. Following the Adomian decomposition analysis, we rewrite (1) in the following operator form:

$$\begin{aligned} L_z\Psi = {}& \lambda \mathrm{i} L_{2,t}\Psi + \delta \mathrm{i}|\Psi|^2\Psi + \varepsilon\left[R_1 L_{3,t}\Psi + R_2 L_{1,t}(|\Psi|^2\Psi) + R_3\Psi L_{1,t}(|\Psi|^2)\right] \\ & + \varepsilon^2 \mathrm{i}\left[R_4 L_{4,t}\Psi + R_5 L_{2,t}(|\Psi|^2\Psi) + R_6 L_{1,t}(\Psi L_{1,t}(|\Psi|^2)) + R_7 L_{1,t}(L_{1,t}\Psi|\Psi|^2)\right]. \end{aligned} \tag{2}$$

Similar to [7], we define for (2) the linear operators $L_z \equiv \frac{\partial}{\partial z}, L_{1,t} \equiv \frac{\partial}{\partial t}, L_{2,t} \equiv \frac{\partial^2}{\partial t^2}, L_{3,t} \equiv \frac{\partial^3}{\partial t^3}$ and $L_{4,t} \equiv \frac{\partial^4}{\partial t^4}$. By defining the onefold right-inverse operator $L_z^{-1} \equiv \int_0^z (\cdot)\mathrm{d}z$, we find that

$$\begin{aligned} \Psi(t,z) = {}& \Psi(t,0) + L_z^{-1}\left\{\lambda \mathrm{i} L_{2,t}\Psi + \delta \mathrm{i}|\Psi|^2\Psi + \varepsilon\left[R_1 L_{3,t}\Psi + R_2 L_{1,t}(|\Psi|^2\Psi) + R_3\Psi L_{1,t}(|\Psi|^2)\right]\right. \\ & \left. + \varepsilon^2 \mathrm{i}\left[R_4 L_{4,t}\Psi + R_5 L_{2,t}(|\Psi|^2\Psi) + R_6 L_{1,t}(\Psi L_{1,t}(|\Psi|^2)) + R_7 L_{1,t}(L_{1,t}\Psi|\Psi|^2)\right]\right\}. \end{aligned} \tag{3}$$

Therefore

$$\begin{aligned} \Psi(t,z) = {}& f(t) + L_z^{-1}\left\{\lambda \mathrm{i} L_{2,t}\Psi + \delta \mathrm{i} G_1(\Psi) + \varepsilon\left[R_1 L_{3,t}\Psi + R_2 G_2(\Psi) + R_3 G_3(\Psi)\right]\right. \\ & \left. + \varepsilon^2 \mathrm{i}\left[R_4 L_{4,t}\Psi + R_5 G_4(\Psi) + R_6 G_5(\Psi) + R_7 G_6(\Psi)\right]\right\}. \end{aligned} \tag{4}$$

The decomposition method suggests that the linear terms $\Psi(t,z)$ be decomposed by an infinite series of components

$$\Psi(t,z) = \sum_{n=0}^{\infty} \Psi_n(t,z), \tag{5}$$

where the components $\Psi_0, \Psi_1, \Psi_2, \cdots$, as will be seen later, are to be determined individually in an easy way through a recursive relation that involves simple integrals. The nonlinear operators $G_1(t,z), G_2(t,z), \cdots, G_6(t,z)$ are defined by the infinite series

$$G_i(\Psi) = \sum_{n=0}^{\infty} A_{i,n}, \quad i = 1,2,\cdots,6. \tag{6}$$

That means that the nonlinear terms $|\Psi|^2\Psi$, $L_{1,t}(|\Psi|^2\Psi)$, $\Psi L_{1,t}(|\Psi|^2)$, $L_{2,t}(|\Psi|^2\Psi)$, $L_{1,t}(\Psi L_{1,t}(|\Psi|^2))$ and $L_{1,t}(L_{1,t}\Psi|\Psi|^2)$ are represented series of $A_{i,n}$ $(i = 1,2,\cdots,6)$ which are called Adomian polynomials. In next $\Psi_n(t,z)\,(n \geq 0)$ is the component of $\Psi(t,z)$ that will elegantly be determined. Hence, upon substituting these decomposition series into (4) yields

$$\begin{aligned} \sum_{n=0}^{\infty} \Psi_n(t,z) = {}& f(t) + L_z^{-1} \sum_{n=0}^{\infty} \left\{\lambda \mathrm{i} L_{2,t}\Psi_n(t,z) + \delta \mathrm{i} A_{1,n} + \varepsilon\left[R_1 L_{3,t}\Psi_n(t,z) + R_2 A_{2,n} + R_3 A_{3,n}\right]\right. \\ & \left. + \varepsilon^2 \mathrm{i}\left[R_4 L_{4,t}\Psi_n(t,z) + R_5 A_{4,n} + R_6 A_{5,n} + R_7 A_{6,n}\right]\right\}. \end{aligned} \tag{7}$$

The method suggests that the zeroth component Ψ_0 is usually defined as the terms arising from initial conditions. Then we obtain the component series solution by the following recursive relationship:

$$\Psi_0(t,0) = f(t), \tag{8}$$

$$\begin{aligned}\Psi_{n+1} = L_z^{-1}\big\{&\lambda \mathrm{i} L_{2,t}\Psi_n + \delta \mathrm{i} A_{1,n} \\ &+ \varepsilon\big[R_1 L_{3,t}\Psi_n + R_2 A_{2,n} + R_3 A_{3,n}\big] \\ &+ \varepsilon^2 \mathrm{i}\big[R_4 L_{4,t}\Psi_n + R_5 A_{4,n} \\ &\quad + R_6 A_{5,n} + R_7 A_{6,n}\big]\big\},\end{aligned} \tag{9}$$

where $n \geq 0$.

The Adomian polynomials $A_{i,n}$ $(i = 1,2,\cdots,6)$ can be generated for all forms of nonlinearity which are generated according to the following algorithm:

$$A_{i,n} = \frac{1}{n!}\Big[\frac{\mathrm{d}^n}{\mathrm{d}\alpha^n} G_i\Big(\sum_{k=0}^{n} \alpha^k \Psi_k\Big)\Big]_{\alpha=0}, \quad n \geq 0. \tag{10}$$

This formula is easy to be set in a computer code to get as many polynomials as we need in the calculation. For example, we can give the first few Adomian polynomials of the $A_{i,n}$ $(i = 1,2,3)$ as

$$A_{1,0} = \Psi_0|\Psi_0|^2,$$

$$A_{1,1} = 2|\Psi_0|^2\Psi_1 + \Psi_0^2\overline{\Psi}_1,$$

$$A_{1,2} = 2|\Psi_0|^2\Psi_2 + \overline{\Psi}_0\Psi_1^2 + 2\Psi_0|\Psi_1|^2 + \Psi_0^2\overline{\Psi}_2,$$

$$\begin{aligned}A_{1,3} = {}& 2|\Psi_0|^2\Psi_3 + 2\overline{\Psi}_0\Psi_1\Psi_2 + 2\Psi_0\overline{\Psi}_1\Psi_2 + |\Psi_1|^2\Psi_1 \\ & + 2\Psi_0\Psi_1\overline{\Psi}_2 + \Psi_0^2\overline{\Psi}_3,\end{aligned}$$

$$A_{2,0} = \frac{\partial A_{1,0}}{\partial t} = (\Psi_0|\Psi_0|^2)_t,$$

$$A_{2,1} = \frac{\partial A_{1,1}}{\partial t} = \big(2|\Psi_0|^2\Psi_1 + \Psi_0^2\overline{\Psi}_1\big)_t,$$

$$\begin{aligned}A_{2,2} = \frac{\partial A_{1,2}}{\partial t} = {}& \big(2|\Psi_0|^2\Psi_2 + \overline{\Psi}_0\Psi_1^2 \\ & + 2\Psi_0|\Psi_1|^2 + \Psi_0^2\overline{\Psi}_2\big)_t,\end{aligned}$$

$$\begin{aligned}A_{2,3} = \frac{\partial A_{1,3}}{\partial t} = {}& \big(2|\Psi_0|^2\Psi_3 + 2\overline{\Psi}_0\Psi_1\Psi_2 + 2\Psi_0\overline{\Psi}_1\Psi_2 \\ & + |\Psi_1|^2\Psi_1 + 2\Psi_0\Psi_1\overline{\Psi}_2 + \Psi_0^2\overline{\Psi}_3\big)_t,\end{aligned}$$

$$A_{3,0} = \Psi_0|\Psi_0|_t^2,$$

$$A_{3,1} = (|\Psi_0|^2\Psi_1)_t + \Psi_0\big((\Psi_0\overline{\Psi}_1)_t + \overline{\Psi}_{0,t}\Psi_1\big),$$

$$\begin{aligned}A_{3,2} = {}& \big(\Psi_0|\Psi_1|^2 + |\Psi_0|^2\Psi_2\big)_t \\ & + \Psi_0\big((\Psi_0\overline{\Psi}_2)_t + \Psi_1\overline{\Psi}_{1,t} + \overline{\Psi}_{0,t}\Psi_2\big) \\ & + (\overline{\Psi}_0\Psi_1)_t\Psi_1,\end{aligned}$$

$$\begin{aligned}A_{3,3} = {}& \big(\Psi_0\Psi_1\overline{\Psi}_2 + \overline{\Psi}_0\Psi_1\Psi_2 + |\Psi_0|^2\Psi_3\big)_t + \Psi_0(\Psi_0\overline{\Psi}_3)_t \\ & + \Psi_1\overline{\Psi}_{2,t} + \overline{\Psi}_{0,t}\Psi_3 + \big(|\Psi_1|_t^2 + \overline{\Psi}_{0,t}\Psi_2\big)\Psi_1 \\ & + (\Psi_0\Psi_2)_t\overline{\Psi}_1.\end{aligned}$$

The rest of the polynomials can be constructed in a similar manner. In case the nonlinear terms $G_1(\Psi) = \Psi|\Psi|^2, G_2(\Psi) = L_{1,t}(\Psi|\Psi|^2)$ and $G_3(\Psi) = \Psi L_{1,t}|\Psi|^2$ are real functions, then the Adomian polynomials are evaluated by first writing

$$|\Psi| = \Psi|H(\Psi) - H(-\Psi)|, \tag{11}$$

where $H(u)$ is the Heaviside (step) function. Hence, (11) yields

$$G_1(\Psi) = |\Psi|^2\Psi = \Psi^3[H(\Psi) - H(-\Psi)]^2,$$

$$G_2(\Psi) = (\Psi^3)_t[H(\Psi) - H(-\Psi)]^2,$$

$$G_3(\Psi) = \frac{2}{3}(\Psi^3)_t[H(\Psi) - H(-\Psi)]^2.$$

Therefore,

$$G_1(\Psi) = \sum_{n=0}^{\infty}[H(\Psi) - H(-\Psi)]^2 A'_{1,n}\{\Psi^3\},$$

$$\begin{aligned}G_3(\Psi) &= \frac{2}{3}G_2\Psi \\ &= \sum_{n=0}^{\infty}[H(\Psi) - H(-\Psi)]^2 A'_{2,n}\{(\Psi^3)_t\},\end{aligned}$$

where $A'_{1,n}\{\Psi^3\}$ and $A'_{2,n}\{(\Psi^3)_t\}$ are the Adomian polynomials given by

$$A'_{1,0} = \Psi_0^3, \quad A'_{1,1} = 3\Psi_0^2\Psi_1,$$

$$A'_{1,2} = 3\Psi_0^2\Psi_2 + 3\Psi_0\Psi_1^2,$$

$$A'_{1,3} = 3\Psi_0^2\Psi_3 + 6\Psi_0\Psi_1\Psi_2 + \Psi_1^3,$$

$$A'_{2,0} = \Psi_{0,t}^3, \quad A'_{2,1} = 3(\Psi_0^2\Psi_1)_t,$$

$$A'_{2,2} = 3(\Psi_0^2\Psi_2 + \Psi_0\Psi_1^2)_t,$$

$$A'_{2,3} = (3\Psi_0^2\Psi_3 + 6\Psi_0\Psi_1\Psi_2 + \Psi_1^3)_t.$$

The function $G_i(\Psi)$ $(i = 1,2)$ is piecewise differentiable with a singularity at the origin. Since $[H(\Psi) - H(-\Psi)]^2 = 1$ for $\Psi \neq 0$, it follows from (11) that, if we avoid the origin, $G_1(\Psi) = \Psi^3 = \sum_{n=0}^{\infty} A'_{1,n}$ and $G_2(\Psi) = (\Psi^3)_t = \sum_{n=0}^{\infty} A'_{2,n}$.

3. Exemplification of the ADM

We first consider the application of the decomposition method to the HDCNS equation (1) with the initial condition

$$\Psi(t,0) = E\,\mathrm{cn}(\sqrt{a}t,m)\mathrm{e}^{\mathrm{i}Kt}, \tag{12}$$

where

$$\begin{aligned} a &= \frac{3K^2(5R_5+4R_6+R_7)}{(2m^2-1)(3R_5+2R_6+R_7)} \\ &\quad + \frac{3K(R_4R_2-2R_6R_1-3R_5R_1-R_7R_1)}{\varepsilon R_4(2m^2-1)(3R_5+2R_6+R_7)} \\ &\quad + \frac{3\delta}{\varepsilon^2(2m^2-1)(3R_5+2R_6+R_7)} \\ &\quad - \frac{\lambda}{\varepsilon^2(2m^2-1)R_4}, \\ E &= \sqrt{\frac{6am^2R_4}{3R_5+2R_6+R_7}}, \\ K &= \frac{2R_6R_1-3R_4R_2-2R_4R_3+R_7R_1+3R_5R_1}{6\varepsilon R_4(R_5+R_6)}, \end{aligned} \tag{13}$$

and $\mathrm{cn}(\sqrt{a}t,m)$ is the Jacobian elliptic cosine function and m $(0<m<1)$ a module of the Jacobian elliptic function.

Applying the inverse operator L_z^{-1} on both sides of (2) and using the initial condition (12) and the decomposition series (5) and (6) yields

$$\begin{aligned} \sum_{n=0}^{\infty}\Psi_n(t,z) &= E\,\mathrm{cn}(\sqrt{a}t,m)\mathrm{e}^{\mathrm{i}Kt} + L_z^{-1}\sum_{n=0}^{\infty}\big\{\lambda\mathrm{i}L_{2,t}\Psi_n(t,z) \\ &+ \delta\mathrm{i}A_{1,n} + \varepsilon\big[R_1L_{3,t}\Psi_n(t,z)+R_2A_{2,n}+R_3A_{3,n}\big] \\ &+ \varepsilon^2\mathrm{i}\big[R_4L_{4,t}\Psi_n(t,z)+R_5A_{4,n}+R_6A_{5,n}+R_7A_{6,n}\big]\big\}. \end{aligned} \tag{14}$$

Proceeding as before, the Adomian decomposition method gives the recurrence relation

$$\Psi_0 = \Psi(t,0) = E\,\mathrm{cn}(\sqrt{a}t,m)\mathrm{e}^{\mathrm{i}Kt}, \tag{15}$$

$$\begin{aligned} \Psi_{n+1} &= L_z^{-1}\big\{\lambda\mathrm{i}L_{2,t}\Psi_n + \delta\mathrm{i}A_{1,n} \\ &+ \varepsilon\big[R_1L_{3,t}\Psi_n + R_2A_{2,n} + R_3A_{3,n}\big] \\ &+ \varepsilon^2\mathrm{i}\big[R_4L_{4,t}\Psi_n + R_5A_{4,n} + R_6A_{5,n} + R_7A_{6,n}\big]\big\}, \end{aligned} \tag{16}$$

where $n \geq 0$. The resulting components are

$$\Psi_0 = E\,\mathrm{cn}(\sqrt{a}t,m)\mathrm{e}^{\mathrm{i}Kt},$$

$$\begin{aligned} \Psi_1 &= E\big[\mathrm{i}\Omega\,\mathrm{cn}(\sqrt{a}t,m) \\ &\quad - \sqrt{a}b\,\mathrm{sn}(\sqrt{a}t,m)\mathrm{dn}(\sqrt{a}t,m)\big]\mathrm{e}^{\mathrm{i}Kt}, \end{aligned}$$

$$\Psi_2 = E\big[-\frac{1}{2}(\Omega^2+ab^2)\mathrm{cn}(\sqrt{a}t,m) + m^2ab^2\mathrm{sn}^2(\sqrt{a}t,m)\mathrm{cn}(\sqrt{a}t,m) - \mathrm{i}\sqrt{a}b\Omega\,\mathrm{sn}(\sqrt{a}t,m)\mathrm{dn}(\sqrt{a}t,m)\big]\mathrm{e}^{\mathrm{i}Kt},$$

$$\begin{aligned} \Psi_3 &= E\big[-\frac{1}{2}\mathrm{i}\Omega(\Omega^2+ab^2)\mathrm{cn}(\sqrt{a}t,m) - m^2a^{3/2}b^3\mathrm{sn}^3(\sqrt{a}t,m)\mathrm{dn}(\sqrt{a}t,m) \\ &\quad + \mathrm{i}\Omega m^2ab^2\mathrm{sn}^2(\sqrt{a}t,m)\mathrm{cn}(\sqrt{a}t,m) + \frac{1}{6}\sqrt{a}b(4m^2ab^2+ab^2+3\Omega^2)\mathrm{sn}(\sqrt{a}t,m)\mathrm{dn}(\sqrt{a}t,m)\big]\mathrm{e}^{\mathrm{i}Kt}, \end{aligned}$$

$$\begin{aligned} \Psi_4 &= E\big[\frac{1}{24}(\Omega^4+6\Omega^2ab^2+a^2b^4+4m^2a^2b^4)\mathrm{cn}(\sqrt{a}t,m) + m^4a^2b^4\mathrm{sn}^4(\sqrt{a}t,m)\mathrm{cn}(\sqrt{a}t,m) \\ &\quad - \mathrm{i}\Omega m^2a^{3/2}b^3\mathrm{sn}^3(\sqrt{a}t,m)\mathrm{dn}(\sqrt{a}t,m) - \frac{1}{6}m^2ab^2(3\Omega^2+5ab^2+2m^2ab^2)\mathrm{sn}^2(\sqrt{a}t,m)\mathrm{cn}(\sqrt{a}t,m) \\ &\quad + \frac{1}{6}\mathrm{i}\Omega\sqrt{a}b(4m^2ab^2+\Omega^2+ab^2)\mathrm{sn}(\sqrt{a}t,m)\mathrm{dn}(\sqrt{a}t,m)\big]\mathrm{e}^{\mathrm{i}Kt}, \end{aligned}$$

…,
where

$$\begin{aligned} b &= 4\varepsilon^2R_4K^3 - 3\varepsilon R_1K^2 + 2(2\varepsilon^2aR_4 - 4\varepsilon^2am^2R_4 - \lambda)K + a\varepsilon R_1(2m^2-1), \\ \Omega &= \varepsilon^2R_4K^4 - \varepsilon R_1K^3 + (6\varepsilon^2aR_4 - 12\varepsilon^2am^2R_4 - \lambda)K^2 \\ &\quad + 3a\varepsilon R_1(2m^2-1)K + a(2m^2-1)(2\varepsilon^2am^2R_4 - \varepsilon^2aR_4 + \lambda), \end{aligned} \tag{17}$$

and $\mathrm{sn}(\sqrt{a}t,m)$ and $\mathrm{dn}(\sqrt{a}t,m)$ are the Jacobian elliptic sine function and the third Jacobian elliptic function, respectively. The other components of the decomposition series (5) can be determined in a similar way. Substituting above $\Psi_0,\Psi_1,\Psi_2,\cdots$ into the decomposition series (5), which is a Taylor series, we obtain the closed form of the Jacobian elliptic cosine function solution

$$\begin{aligned}\Psi &= \Psi_0+\Psi_1+\Psi_2+\Psi_3+\Psi_4+\cdots \\ &= E\,\mathrm{cn}(\sqrt{a}(t+bz),m)\mathrm{e}^{\mathrm{i}(Kt+\Omega z)}\end{aligned} \tag{18}$$

with the constants E,a,K and b,Ω are expressed by (13) and (17), respectively.

From these values of the pulse parameters, it is simple to see that both the amplitude and the width of the soliton are uniquely determined from the characteristics of the nonlinear medium, i. e. the dispersion coefficients and the nonlinear coefficients.

Rewriting (1) for the initial condition

$$\Psi_0(t,0) = E_1\mathrm{sn}(\sqrt{a_1}t,m)\mathrm{e}^{\mathrm{i}K_1 t}, \tag{19}$$

where

$$\begin{aligned}a_1 &= -\frac{3K_1^2(5R_5+4R_6+R_7)}{(1+m^2)(3R_5+2R_6+R_7)} \\ &\quad -\frac{3K_1(R_4R_2-2R_6R_1-3R_5R_1-R_7R_1)}{\varepsilon R_4(1+m^2)(3R_5+2R_6+R_7)} \\ &\quad -\frac{3\delta}{\varepsilon^2(1+m^2)(3R_5+2R_6+R_7)} \\ &\quad +\frac{\lambda}{\varepsilon^2(1+m^2)R_4}, \\ E_1 &= \sqrt{-\frac{6a_1m^2R_4}{3R_5+2R_6+R_7}}, \\ K_1 &= \frac{2R_6R_1-3R_4R_2-2R_4R_3+R_7R_1+3R_5R_1}{6\varepsilon R_4(R_5+R_6)},\end{aligned} \tag{20}$$

in a operator form as (2), then using (8) and (9) with (10), one can also construct the terms of the decomposition series. Performing the calculations in (19) and (16) with (10) using Maple and substituting them into (5) gives the Jacobian elliptic sine function solution

$$\Psi = E_1\mathrm{sn}(\sqrt{a_1}(t+b_1z),m)\mathrm{e}^{\mathrm{i}(K_1t+\Omega_1 z)}, \tag{21}$$

where

$$\begin{aligned}b_1 &= 4\varepsilon^2R_4K_1^3-3\varepsilon R_1K_1^2+2(2\varepsilon^2a_1R_4+2\varepsilon^2a_1m^2R_4-\lambda)K_1-a_1\varepsilon R_1(1+m^2), \\ \Omega_1 &= \varepsilon^2R_4K_1^4-\varepsilon R_1K_1^3+(6\varepsilon^2a_1R_4+6\varepsilon^2a_1m^2R_4-\lambda)K_1^2 \\ &\quad -3a_1\varepsilon R_1(1+m^2)K_1+a_1(1+m^2)(\varepsilon^2a_1m^2R_4+\varepsilon^2a_1R_4-\lambda).\end{aligned} \tag{22}$$

When the module $m\to 1$, we can derive the bright solitary wave solution and the dark solitary wave solution from (18) and (21), respectively, i. e.

$$\Psi = E\,\mathrm{sech}[\sqrt{a}(t+bz)]\mathrm{e}^{\mathrm{i}(Kt+\Omega z)} \tag{23}$$

with the constants E,a,K and b,Ω are expressed by

$$\begin{aligned}a &= \frac{3K^2(5R_5+4R_6+R_7)}{(3R_5+2R_6+R_7)} \\ &\quad +\frac{3K(R_4R_2-2R_6R_1-3R_5R_1-R_7R_1)}{\varepsilon R_4(3R_5+2R_6+R_7)} \\ &\quad +\frac{3\delta}{\varepsilon^2(3R_5+2R_6+R_7)}-\frac{\lambda}{\varepsilon^2R_4}, \\ E &= \sqrt{\frac{6aR_4}{3R_5+2R_6+R_7}}, \\ K &= \frac{2R_6R_1-3R_4R_2-2R_4R_3+R_7R_1+3R_5R_1}{6\varepsilon R_4(R_5+R_6)}, \\ b &= 4\varepsilon^2R_4K^3-3\varepsilon R_1K^2-2(2\varepsilon^2aR_4+\lambda)K+a\varepsilon R_1, \\ \Omega &= \varepsilon^2R_4K^4-\varepsilon R_1K^3-(6\varepsilon^2aR_4+\lambda)K^2 \\ &\quad +3a\varepsilon R_1K+a(\varepsilon^2aR_4+\lambda),\end{aligned}$$

and

$$\Psi = E_1\tanh[\sqrt{a_1}(t+b_1z)]\mathrm{e}^{\mathrm{i}(K_1t+\Omega_1 z)}, \tag{24}$$

with the constants E_1,a_1,K_1 and b_1,Ω_1 are expressed by

$$\begin{aligned}a_1 &= -\frac{3K_1^2(5R_5+4R_6+R_7)}{2(3R_5+2R_6+R_7)} \\ &\quad -\frac{3K_1(R_4R_2-2R_6R_1-3R_5R_1-R_7R_1)}{2\varepsilon R_4(3R_5+2R_6+R_7)}\end{aligned}$$

$$-\frac{3\delta}{2\varepsilon^2(3R_5+2R_6+R_7)}+\frac{\lambda}{2\varepsilon^2 R_4},$$

$$E_1=\sqrt{-\frac{6a_1R_4}{3R_5+2R_6+R_7}},$$

$$K_1=\frac{2R_6R_1-3R_4R_2-2R_4R_3+R_7R_1+3R_5R_1}{6\varepsilon R_4(R_5+R_6)},$$

$$b_1=4\varepsilon^2R_4K_1^3-3\varepsilon R_1K_1^2+2(4\varepsilon^2a_1R_4-\lambda)K_1-2a_1\varepsilon R_1,$$

$$\Omega_1=\varepsilon^2R_4K_1^4-\varepsilon R_1K_1^3+(12\varepsilon^2a_1R_4-\lambda)K_1^2-6a_1\varepsilon R_1K_1+2a_1(2\varepsilon^2a_1R_4-\lambda).$$

4. Numerical Results of the ADM

For numerical comparison purposes, we construct the solution $\Psi(t,z)$ as

$$\Psi(t,z)=\lim_{N\to\infty}\psi_N, \tag{25}$$

where $\psi_N(t,z)$, the N-term approximation for $\Psi(t,z)$, is a finite series defined as

$$\psi_N(t,z)=\sum_{n=0}^{N-1}\Psi_n(t,z),\quad N\geq 1, \tag{26}$$

and the recurrence relation is given as in (9) with (10). It is worth pointing out that the advantage of the decomposition methodology is the fast convergence of the solutions in real physical problems. The theoretical treatment of convergence of the decomposition method has been considered in the literature [8, 9]. The obtained results about the speed of convergence of this method were enabling us to solve linear and nonlinear functional equations. In a recent work Ngarhasta et al. [15] have proposed a new approach of convergence of the decomposition series. The authors gave a new condition for obtaining convergence of the decomposition series to the classical presentation of the ADM.

Moreover, as the decomposition method does not require discretization of the variables time and space, it is not effected by computation round off errors and one is not faced with the necessity of large computer memory and time. The accuracy of the decomposition method for (1) is controllable and absolute errors $|\Psi-\psi_N|$ are very small with the present choice of t and z which are given in Table 1. In most cases ψ_N is accurate for quite low values of N (Fig. 1). For an initial condition, we achieve a very good approximation to the partial exact solution by using only 30 terms of the decomposition series, which shows that the speed of convergence of this method is very fast. It is evident that the overall errors can be made smaller by adding new terms of the decomposition series. Both the exact results and the approximate solutions obtained by using the formula (23) are plotted in Fig. 2 for (1). It is evident that, when computing more terms for the decomposition series, the numerical results are getting much more closer to the corresponding exact solutions with the initial condition of (1).

Table 1. Numerical results (in z direction) for modules of approximate result $|\psi_{30}(t,z)|$, exact result $|\Psi(t,z)|$ and absolute error $|\Psi(t,z)-\psi_{30}(t,z)|$, where $\Psi(t,z)=3.343733775\mathrm{sech}(6.687467550z-13.43685610t)\exp[(0.8333333335z-4.322530866t)\mathrm{i}]$ for (1).

(t_i,z_i)	Exact solution $\lvert\Psi\rvert$	Approximate solution $\lvert\psi_{30}\rvert$	Absolute error $\lvert\Psi-\psi_{30}\rvert$
(0.01,0.01)	3.336132141	3.336132140	0.1E-8
(0.02,0.02)	3.313499116	3.313499116	0
(0.02,0.03)	3.335992292	3.335992292	0
(0.03,0.04)	3.313223212	3.313223212	0
(0.05,0.05)	3.161967214	3.161967214	0
(0.04,0.01)	3.004818176	3.004818176	0
(0.05,0.02)	2.911900805	2.911900805	0
(0.04,0.03)	3.162603484	3.162603482	0.2E-8
(0.05,0.04)	3.087852890	3.087852891	0.1E-8
(0.05,0.05)	3.161967214	3.161967214	0

5. Conclusion

In this paper, we considered a numerical treatment for the solution of the HDCNS equation using the ADM. To the best of our knowledge, this is the first result on the application of the ADM to this equation. This method transforms (1) into a recursive relation.

The obtained numerical results compared with the analytical solution show that the method provides remarkable accuracy, especially for small values of the space z. Generally speaking, the ADM provides analytic, verifiable, rapidly convergent approximation which yields insight into the character and the behaviour of the solution just as in the closed form solution. It solves nonlinear problems without requiring linearization, perturbation, or unjustified assumptions which may change the problem being solved. The method can also easily be extended to other similar physical equations, with the aid of Maple (or Matlab, Mathematica, etc.), the course of solving nonlinear evaluation equations can be carried out in a computer.

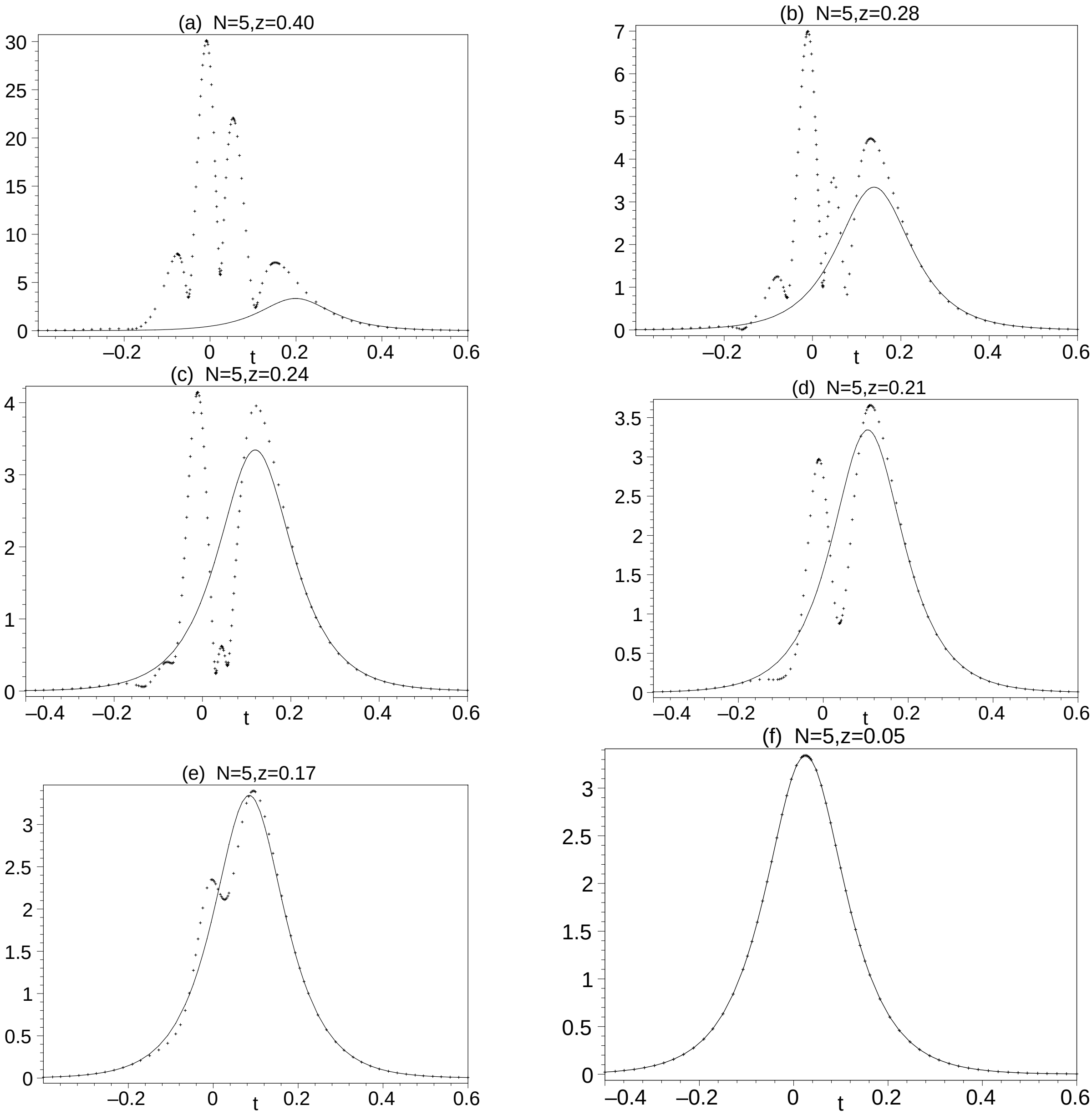

Fig. 1. Module graph of the exact (line) result (23) $|\Psi|$ and approximate (dots) result $|\psi_5|$ of (1) at (a) $z = 0.40$; (b) $z = 0.28$; (c) $z = 0.24$; (d) $z = 0.21$; (e) $z = 0.17$; (f) $z = 0.05$.

As we known, although the decomposition series (5) obtained by using the ADM is infinite, we often replace the exact solution with a finite series

$$\psi_N(t,z) = \sum_{n=0}^{N-1} \Psi_n(t,z),$$

which is quickly convergent towards the accurate solution for quite low values of N. On this account, there is a common phenomenon in the related literature [8 – 14]. It can easily be noted that, no matter whether the examples are from the related literature or from this paper, the space or time variables in the pictures are all taken in small scales. Since the Taylor series method provides the same answer obtained by the ADM, we can proceed from the nature of the Taylor series [16, 17] to study this phenomenon. The Taylor

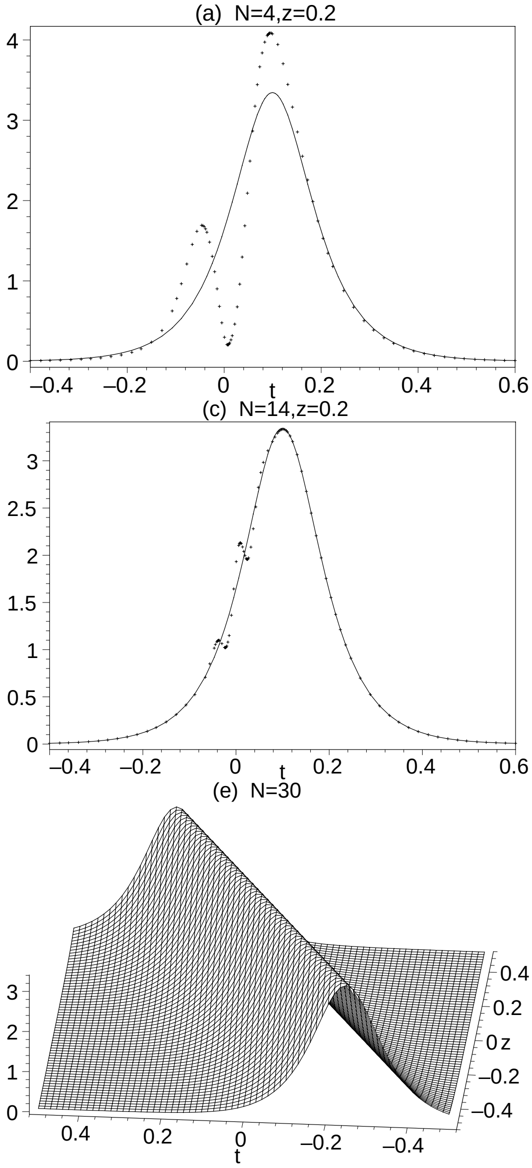

Fig. 2. (a) – (d) Module graphs of the exact (line) result (23) $|\Psi|$ and approximate (dots) result $|\psi_n|$ at (a) $N=4, z=0.2$; (b) $N=8, z=0.2$; (c) $N=14, z=0.2$; (d) $N=30, z=0.2$. (e) The surface shows the module of numerical result ψ_{30} for $-0.5 \le t \le 0.5$ and $-0.5 \le z \le 0.5$.

series expansion of the function $\Psi(t,z)$ about $z=z_0$ is given by

$$\Psi(t,z) = \sum_{n=0}^{\infty} \frac{\Psi^{(n)}(t,z_0)}{n!}(z-z_0)^n \tag{27}$$

or, equivalently,

$$\Psi(t,z) = \sum_{n=0}^{N-1} \frac{\Psi^{(n)}(t,z_0)}{n!}(z-z_0)^n + R_{N-1}, \ N \ge 1. \tag{28}$$

Here, R_{N-1} is a remainder term known as the Lagrange remainder, which is given by

$$\begin{aligned} R_{N-1} &= \underbrace{\int \cdots \int_0^z}_{N} \Psi^{(N)}(t,z)(\mathrm{d}z)^N \\ &= \frac{(z-z_0)^N}{(N!)} \Psi(N)(z^*), \quad z^* \in [z_0, z]. \end{aligned} \tag{29}$$

As we known, the decomposition series (5) is exactly a Taylor series of exact solution Ψ about a point $z=0$, that is

$$\psi_N = \sum_{n=0}^{N-1} \Psi_n(t,z) = \sum_{n=0}^{N-1} \frac{\Psi^{(n)}(t,0)}{n!} z^n, \quad N \geq 1.$$

Then the remainder term R_{N-1}, i. e. the error between analytical and approximate solutions, is

$$R_{N-1} = \Psi - \sum_{n=0}^{N-1} \frac{\Psi^{(n)}(t,0)}{n!} z^n = \Psi - \varphi_N$$

$$= \underbrace{\int \cdots \int_0^z}_{N} \Psi^{(N)}(t,z)(\mathrm{d}z)^N = \frac{(z)^N}{N!} \Psi^{(N)}(z^*),$$

$$z^* \in [0,z], \quad N \geq 1,$$

which can be obtained by using the mean-value theorem. From the formula above, we know that the greater the fetching value of $|z|$ (z is farther and farther from the point $z = 0$) is, the greater is the error R_{N-1} (see Fig. 1), although the approximated solution can be calculated for any t and z. Nearing zero for z, the approximate solution is almost according to the exact solution at any value of time t. From Figs. 1 and 2, one might find out that both the term number N and the value of z influence the approximation precision of the numerical solution (26) for the corresponding exact solution of the HDCNS equation, whereas time t has only a little effect on this.

Acknowledgements

The author is in debt to Profs. J. F. Zhang and C. L. Zheng and Drs. J. M. Zhu and W. H. Huang for their helpful suggestions and fruitful discussions. The work was supported by the National Natural Science Foundation of China (No. 10272071), the Natural Science Foundation of Zhejiang Province of China (No. Y606049) and the Scientific Research Fund of Zhejiang Provincial Education Department of China (No. 20051356).

[1] D. H. Lou, Wave Motion **33**, 339 (2001).
[2] K. B. Dysthe, Proc. R. Soc. London A **369**, 105 (1979).
[3] A. Hasegawa and Y. Kodama, Solitons in Optical Communication, Clarendon Press, Oxford 1995.
[4] G. Adomian, Appl. Math. Comput. **88**, 127 (1997).
[5] S. A. Khuri, Appl. Math. Comput. **97**, 251 (1998).
[6] B. K. Datta, Comput. Math. Appl. **20**, 61 (1990).
[7] S. M. El-Sayed and D. Kaya, Appl. Math. Comput. **172**, 1315 (2006).
[8] G. Adomian and R. Rach, Appl. Math. Comput. **57**, 61 (1993).
[9] A. M. Wazwaz, Partial Differential Equations: Methods and Applications, Balkema Press, Rotterdam 2002.
[10] R. Rach, J. Math. Anal. Appl. **128**, 480 (1987).
[11] J. Y. Edwards, J. A. Roberts, and N. J. Ford, Technical Report No. 309, Manchester Center of Computational Mathematics, Manchester 1997.
[12] A. M. Wazwaz, Appl. Math. Comput. **97**, 37 (1998).
[13] S. M. El-Sayed and M. R. Abdel-Aziz, Appl. Math. Comput. **136**, 151 (2003).
[14] T. Geyikli and D. Kaya, Appl. Math. Comput. **169**, 146 (2005).
[15] N. Ngarhasta, B. Some, K. Abbaoui, and Y. Cherruault, Kybernetes **31**, 61 (2002).
[16] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th ed., Dover, New York 1972.
[17] G. Arfken, Taylor's Expansion, in: Mathematical Methods for Physicists, 3rd ed., Academic Press, Orlando, FL 1985.